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ANALYTIC SOLUTIONS FOR THE PROBLEM OF CONDENSATION OF A TWO-COMPONENT GAS MIXTURE

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For describing the process of condensation of a two-component gas mixture on a cryopanel, a mathematical model is proposed, for which analytic solutions are obtained under various proposed simplifications. Numerically computed solutions are compared with experimental results.

1. Investigations of the process of condensation of a two-component gas mixture into the liquid phase have been made in a series of studies [1-5]. Experimental studies have, for example, been described in [1, 3, 4]. Based on these, the qualitative behavior of the system has been studied and empirical formulae proposed.

In [2], a study was made of gravity-flow film condensation, in which the liquid condensate flowed down an inclined cooled surface and was removed from the system. This enables a study to be made of the steady-state working of such a system, and the construction of a self-modeling or numerical solution of the boundary value problem. In [5], a similar problem is considered on the basis of correlations deriving from the conservation laws. The solution is less accurate, but the basic qualitative rules can be traced from it. In these investigations, however, a steady-state process was studied, in which the concentration field, temperature and other parameters of the boundary layer were time independent. In practice, on the cryopanels of cryogenic condensation pumps, cooled to 15-17 K, there is formed during the condensation of gas mixtures a liquid film precipitating on the cryodeposit. The layer of condensate grows continuously, which brings about an important change with time in the remaining characteristics of the system. Therefore, it is necessary to construct a nonstationary model for describing the process.

2. The construction of a mathematical model of such a process, describing its dynamics for a wide range of input parameters (system pressure, temperature head, overall running time of the process), must in its implementation take into consideration free convection in the gaseous region [2]. If, however, the temperature head in the system (temperature difference between that of the gas far removed from the cryopanel T_∞ and that of the cryopanel T_p is sufficiently great then the contribution from free convection may be neglected, and we can limit ourselves to a spatially uniform model.

For describing the processes of heat and mass transport in the gaseous region, we take a system of coordinates, the origin of which coincides with the boundary of the phase interface (Oy-axis in Fig. 1). Although such a system is not static ($h(\tau)$ being the law of motion

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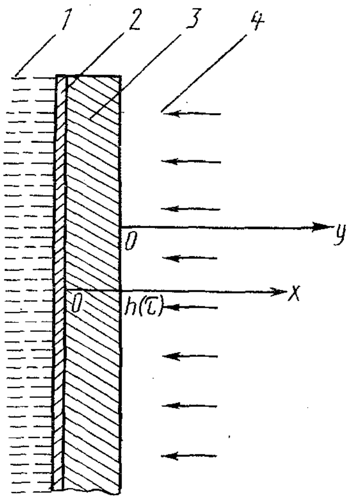

Fig. 1. Geometric configuration of the process: 1) cooling agent; 2) cryopanel; 3) cryodeposit; 4) two-component gas.

of the phase interface boundary as a function of time), this may be neglected, since obviously the velocity of this system $\left|\frac{dh}{d\tau}\right| \ll |v|$, where v is the velocity of the gas. As will be seen later, the relationship between $\frac{dh}{d\tau}$ and v is described by the ratio of the densities of gas and condensate, these differing by several orders of magnitude. We note that in the case of condensation of a freon mixture, the density ratio in the worst case may reach 0.05. If necessary, the motion of the coordinate system may be taken into account with the introduction in the gaseous region of an additional convective term.

The growth of the condensate layer is slower than the transfer processes established in the gaseous region. Therefore, in describing heat and mass transfer in the gaseous region we shall limit ourselves to steady state equations:

$$vT_{y,\mathrm{g}} = a_{\mathrm{g}} T_{yy,\mathrm{g}} \quad y > 0; \quad vW_{y,\mathrm{g}} = DW_{yy\mathrm{g}}, \quad y > 0;$$
$$T_{\mathrm{g}}|_{y=0} = T_{\mathrm{b,g}} \quad W_{\mathrm{g}}|_{y=0} = W_{\mathrm{b.g}}; \quad T_{\mathrm{g}}|_{y=\infty} = T_\infty, \quad W_{\mathrm{g}}|_{y=\infty} = W_\infty. \tag{1}$$

The solution of equations (1), if v < 0 (the gas is moving in the direction of the cryopanel) is:

$$T_{\mathrm{g}}(y) = T_\infty - (T_\infty - T_{\mathrm{b,g}}) \exp\left(\frac{v}{a_{\mathrm{g}}} y\right), \tag{2}$$

$$W_{\mathrm{g}}(y) = W_\infty - (W_\infty - W_{\mathrm{b,g}}) \exp\left(\frac{v}{D} y\right). \tag{3}$$

In Eqs. (2) and (3), there is a nonexplicit time dependence; the values v, $T_{\mathrm{b,g}}$, $W_{\mathrm{b,g}}$, generally speaking depend on τ.

Let us consider the question of the boundary conditions on the moving phase boundary. Convective and diffusive flows (of the first and second gas components) towards the condensate (negative direction of axis Oy) are such that [6]:

$$j_{1,\mathrm{conv}} = -\rho_{\mathrm{g}} W_{\mathrm{g}}|_{y=0} v, \quad j_{2,\mathrm{conv}} = -\rho_{\mathrm{g}}(1 - W_{\mathrm{g}}|_{y=0})\, v,$$
$$j_{1,\mathrm{diff}} = \rho_{\mathrm{g}} D \left.\frac{dW_{\mathrm{g}}}{dy}\right|_{y=0}, \quad j_{2,\mathrm{diff}} = \rho_{\mathrm{g}} D \frac{\partial}{\partial y}(1 - W_{\mathrm{g}}|_{y=0}) = -\rho_{\mathrm{g}} D \left.\frac{\partial W_{\mathrm{g}}}{dy}\right|_{y=0}. \tag{4}$$

From which it follows that the total flow of matter in the direction of the condensate is $j = -\rho_{\mathrm{g}} v$, and the law of conservation of matter at the phase interface boundary takes the form $\rho_{\mathrm{g}} dh = j d\tau$, or

$$\frac{dh}{d\tau} = -\frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{c}}} v. \tag{5}$$

The law of preservation of the first component may be written in the form $\rho_c W_c dh = (j_{1,conv} + j_{1,diff})d\tau$, which, from (4) and (5) gives the equation

$$(W_{b,g} - W_{b,c})v = D\left.\frac{dW_g}{dy}\right|_{y=0}. \qquad (6)$$

We note that the law of conservation of the second component is fulfilled automatically, being a consequence of (5) and (6).

Let us write the law of conservation of heat energy. Let λ_g and c_g be respectively the thermal capacity of the two-component gas mixture. These, we shall consider to be independent of temperature. Then the heat flow to the condensate on the phase interface boundary is

$$q_g = -c_g\rho_g v T_g|_{y=0} + \lambda_g\left.\frac{\partial T_g}{\partial y}\right|_{y=0}. \qquad (7)$$

If q_c is the heat flow in the condensate in the direction of the cryopanel on the phase interface boundary, then the law of conservation of heat energy yields the equation

$$q_g - q_c = \frac{dh}{d\tau}(c_c\rho_c T_b - r\rho_c). \qquad (8)$$

The RHS of this equation denotes the quantity of heat in the volume of matter condensed in time $d\tau$, and the quantity of heat given out on account of the phase transition. We can transform (8), by taking account of (6) and (5), to the form

$$\lambda_g\left.\frac{\partial T_g}{\partial y}\right|_{y=0} - q_c = -\rho_g v[(c_c - c_g)T_b - r]. \qquad (9)$$

Still one more condition at the phase interface boundary is obtained from the following considerations. We shall reckon that for each moment of time τ, matter on the phase interface boundary is in a state of thermodynamic equilibrium (quasi-static process), and the compositions of the gas and condensate on the phase interface boundary ($W_{b,g}$, $W_{b,c}$), are determined by the conditions of the phase equilibrium by the value T_b:

$$W_{b,c} = f_1(T_b),\ W_{b,g} = f_2(T_b). \qquad (10)$$

A typical form of the functions f_1^{-1} and f_2^{-1} inverse to the above is given in [2].

Let us compare the form of solution (3) for $W_g(y)$ and condition (6). After substituting (3) in (6), remembering that $v \neq 0$, we get the result

$$W_{b,c} = W_\infty, \qquad (11)$$

which means that, with the assumption made, there occurs so-called full condensation, in which the composition of the condensate is the same as that of the gas distant from the cryopanel.

In [2] it has been shown how the temperature of the phase interface boundary T_b and the value $W_{b,g}$ are determined by W_∞. Analytically, this problem reduced to a solution of the equation $W_\infty = f_1(T_b) \Rightarrow T_b$, and, further, to computing $W_{b,c} = f_2(T_b)$. Therefore, in the framework of the model, T_b and $W_{b,g}$ are independent of time, and are determined by the initial data.

The process of heat diffusion in the condensate is described by the boundary-value problem for the thermal conductivity equation with constant coefficients

$$T_{\tau,c} = a_c T_{xx,c},\ \tau > 0,\ 0 < x < h(\tau), \qquad (12)$$
$$T_c|_{x=0} = T_p,\ T_c|_{x=h(\tau)} = T_b,\ T_c|_{\tau=0} = T_{0,c}(x),\ 0 < x < h(0).$$

where, from Fig. 1, the point x = 0 of axis Ox coincides with the surface of the cryopanel, at temperature T_p. The heat flow in the condensate q_c, appearing in (8) is expressed in terms of the field T_c:

$$q_c = \lambda_c\left.\frac{\partial T_c}{\partial x}\right|_{x=h(\tau)}. \qquad (13)$$

Equations (2), (7) and (5) enable us to transform (8) to the following from (noting that $a_g = \frac{\lambda_g}{c_g \rho_g}$):

$$\lambda_g \frac{\partial T_c}{\partial x}\bigg|_{x=h(\tau)} = \rho_c \frac{dh}{d\tau}[r + c_g T_\infty - c_g T_b]. \tag{14}$$

Therefore, with the assumptions made, the problem reduced to solving Eqs. (12) and (14) in which there is a moving boundary.

3. Let us make in Eqs. (12), (14) a change of variables $\eta = x/h(\tau)$, often used in problems with moving boundaries. Then we get

$$\left(a_c = \frac{\lambda_c}{\rho_c c_c}\right):$$

$$T_{\tau,c} = \frac{a_c}{h(\tau)} T_{\eta\eta,c} + \frac{\eta}{h(\tau)} \frac{dh}{d\tau} T_{\eta,c} \quad \tau > 0,\ 0 < \eta < 1, \tag{15a}$$

$$T_c|_{\eta=0} = T_p,\ T_c|_{\eta=1} = T_b, \tag{15b}$$

$$\frac{\lambda_c}{h(\tau)} T_{\eta,c}|_{\eta=1} = \rho_c \frac{dh}{d\tau}[r + c_g T_\infty - c_c T_b], \tag{15c}$$

$$T_c|_{\tau>0} = T_0(h(0)\eta,\ 0 < \eta < 1. \tag{15d}$$

If h(0) = 0 then the initial condition (15d) becomes superfluous.

We shall neglect the influence of the specific thermal capacity of the condensate in the new system of coordinates, assuming that the change of field T_c in it with time is sufficiently slow. This means that $T_{\tau,c} \simeq 0$ in (15a), and its solution with condition (15b) has the form

$$T_c(\eta, \tau) = \frac{T_b - T_p}{\int\limits_0^1 \exp\left(-\frac{z}{2a_c}\xi^2\right) d\xi} \int\limits_0^\eta \exp\left(-\frac{z}{2a_c}\xi^2\right) d\xi + T_p. \tag{16}$$

Equation (16) gives the dependence of the solution $T_c(\eta, \tau)$ on the mode of boundary movement and on time, since here $z = h\frac{dh}{d\tau}$. Substituting this expression in the remaining condition (15c), and making simple transformations, we can write:

$$p \exp\left(-\frac{z}{2a_c}\right) = z \int\limits_0^1 \exp\left(-\frac{z}{2a_c}\xi^2\right) d\xi,\quad p = \frac{\lambda_c(T_b - T_p)}{\rho_c[r + c_g T_\infty - c_c T_b]}. \tag{17}$$

In Eqs. (17), the dependence of the variable on time is present only in parametric from $z = h\frac{dh}{d\tau}$; therefore, (17) is the transcendental equation for determining the root $z = z_0 =$ const, and the function $h = h(\tau)$ is found by solving the Cauchy problem:

$$h\frac{dh}{d\tau} = z_0,\ h|_{\tau=0} = h(0) = h_0. \tag{18}$$

In the case $h_0 = 0$, which will be considered below, the solution of (18) is:

$$h(\tau) = \sqrt{2z_0\tau}. \tag{19}$$

4. We can transform Eq. (17), making a change of variables:

$$B = \frac{\sqrt{p}}{\sqrt{2a_c}},\quad s = \frac{\sqrt{z}}{\sqrt{2a_c}};\quad B^2 = se^{s^2}\frac{\sqrt{\pi}}{2}\,\mathrm{erf}\,(s), \tag{20}$$

where $\mathrm{erf}\,(s) = \frac{2}{\sqrt{\pi}} \int\limits_0^s e^{-\xi^2} d\xi$, the probability integral [7].

Clearly, the transcendental equation (20) has a single root $s = s_0$, since the RHS of the third equation (20) is continuous and grows monotonically, tending to zero at $s = 0$, and tending to ∞ at $s \to 0$. This root is easily found graphically, or by using a suitable numerical method.

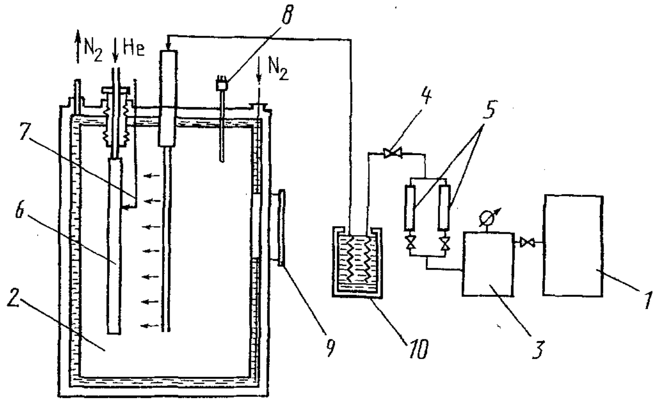

Fig. 2. Schematic of the installation used for condensation of the two-component gas: 1) tank with gas mixture; 2) condensation pump; 3) reservoir; 4) valve; 5) flowmeter; 6) cryopanel; 7) thickness gauge; 8) pressure gauge; 9) observation window; 10) nitrogen heat exchanger.

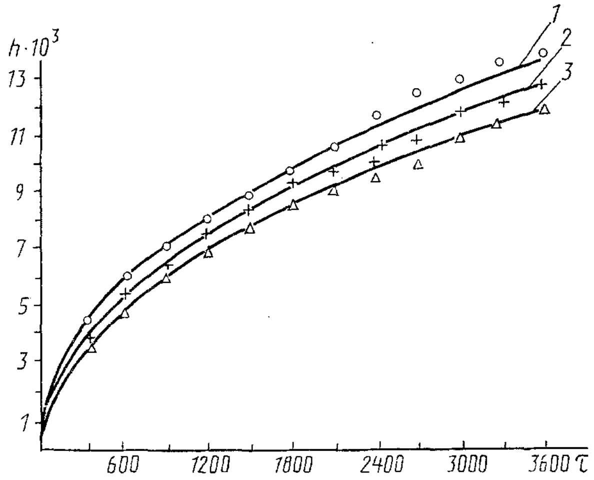

Fig. 3. Time dependence of the thickness of the cryoprecipitate (curves are theoretical; points are experimental): 1) p_g = 3990 Pa, 2) 2660 Pa, 3) 1330 Pa. $h \cdot 10^3$ in meters, τ in sec.

For small B values, however, we can get an expression for $s_0 = s_0(B)$ in the form of an exponential power series. Substituting in (20) the expression erf(s) from [7], Eq. 7.1.6, and after simplification, we get

$$B^2 = s^2 \sum_{n=0}^{n=\infty} \frac{2^n s^{2n}}{(2n+1)!!}. \tag{21}$$

To Eq. (21) can be applied the theorem for the expansion of the roots of a transcendental equation [8], p. 49, from which it follows that there exists a $B_0 > 0$ such that in the circle $|B| < 0$, the function $s_0(B)$ has two simple holomorphic branches. Using the method of undetermined coefficients, we can find any section of the exponential power series for each of them. We are interested in that branch which for real $B > 0$ gives real $s = s_0 > 0$.

The first four terms of the exponential power series of that branch are:

$$s_0 = B - \frac{B^3}{3} + \frac{23}{90} B^5 - \frac{157}{630} B^7 + O(B^9),\ z_0 = 2a_c s_0^2. \tag{22}$$

This representation ensures the necessary accuracy in practice, since usually $B < 0.4$.

5. To check the legitimacy of the assumption of a negligibly small $T_{\tau,c}$, made in solving the problem (15), numerical solution was carried out using a second order accuracy finite-difference approximation in η, and a first order in τ [9]. A comparison of the function h(τ) obtained from Eq. (19) with the numerical solution showed that the maximum deviation did not exceed 3%, and this deviation was mainly caused by error in the numerical

algorithm. An analysis of the temperature field $T_c(\eta, \tau)$ obtained in the numerical experiment shows that it is practically independent of τ (in the variables η, τ, i.e., in a moving system of coordinates), which again confirms the correctness of the assumptions made.

It is interesting that from (19) and (5) it follows that $v = \frac{\rho_c}{\rho_g}\sqrt{\frac{z_0}{2\tau}}$, and as $\tau \to +0$, the velocity of movement of the gas tends to infinity. From the physical point of view, this conclusion is not unexpected, since from the experiment, there is observed initially the so-called jet flow, with a high velocity. This is, however, short-lived.

We note also that for very small B in Eq. (22), we may limit ourselves to taking the first term only, which, as it is not difficult to show, is an equivalent assumption to neglecting the term $T_{\tau,c}$ in the left part of the equation in (12). An analytic solution constructed in this way is bound up with the neglect of the contribution of the moving boundary in (15a), since for small values of B, the role of the second term on RHS of Eq. (15a) is negligible. For large B, this contribution may, however, amount to 5% or more.

Therefore, an attempt to construct an analytic solution deriving directly from (12) may lead to appreciable error.

6. We now return to the experimental results and to a comparison with the results of calculations. In Fig. 2 is shown the schematic of a test installation. The gas mixture of nitrogen and oxygen, of a given composition, from tank 1 entered the condensation pump 2 via the reservoir 3. The gas flow was regulated by valve 4 and measured by flowmeter 5 (type RS-3A or RM-GS-0.25 rotameter), and also by the pressure fall in tank 1. The gas mixture condenses on cryopanel 6; the thickness of the cryoprecipitate layer is measured by the contact thickness gauge 7. Cooling of the cryopanel to 15-17 K was done by circulating gaseous helium. The gas pressure in the pump was measured with types 13VTZ-003 and VDO-1 vacuum gauges, connected to tube 8. The temperatures of the cryopanel, the nitrogen screen and the gas in the pump were measured with copper—iron and copper—constantan thermocouples, with type F-30 voltmeter.

In Fig. 3 are shown experimental and computed curves for different pressures of the gas mixture, consisting of 25% and 75% nitrogen. As seen from the diagram, the difference between theory and experiment did not exceed 5%. This result confirms the assumption made in this work that under experimental conditions, the chief factor determining the mass transfer to the cryopanel is a transverse motion of the gas to the phase boundary interface.

NOTATION

T, temperature; p, pressure; v, gas velocity; h, cryoprecipitate thickness; τ, time; W, mass component; f, mass flow; ρ, density; c, thermal capacity; λ, thermal conductivity; a, temperature conductivity; r, heat of sublimation; D, diffusion coefficient; x, y, current coordinates; P, mathematical parameter; η, dimensionless coordinate inside the condensate; ξ, integration variable; T_τ, T_x, T_y, T_η, W_τ, W_y, derivatives with respect to coordinates and time. Abbreviations: b, phase interface boundary; ∞, distant from phase interface; conv, convective component; diff, diffusion component; c, condensate; g, gas; 0, initial value; p, cryopanel surface; 1, 2, numbers of components.

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